Renormalization group analysis for thermal turbulent transport

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In this study, we continue with our previous renormalization group analysis of incompressible turbulence, aiming at determination of various thermal transport properties. In particular, the temperature field T is considered a passive scalar. The quasinormal approximation is assumed for the statistical correlation between the velocity and temperature fields. A differential argument leads to derivation of the turbulent Prandtl number Pr_t as a function of the turbulent Peclet Pe_t number, which in turn depends on the turbulent eddy viscosity ν_t. The functional relationship between Pr_t and Pe_t is comparable to that of Yakhot *et al.* [Int. J. Heat Mass Transf. **30**, 15 (1987)] and is in close consistency with direct-numerical-simulation results as well as measured data from experiments. The study proceeds further with limiting the operation of renormalization group analysis, yielding an inhomogeneous ordinary differential equation for an invariant thermal eddy diffusivity σ. Simplicity of the equation renders itself a closed-form solution of σ as a function of the wave number k, which, when combined with a modified Batchelor's energy spectrum for the passive temperature T, facilitates determination of the Batchelor constant C_B and a parallel Smagorinsky model and the model constant C_P for thermal turbulent energy transport.

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I. INTRODUCTION

Presently, we continue to pursue recursive renormalization group (RG) analysis [1] of incompressible turbulence, aiming at determining various turbulent transport properties. The analysis starts with the Naviver-Stokes equation in the wave number space under the hypothesis that large-scale eddies are statistically independent of those of smaller scales. In a previous study, we were able to determine several turbulent flow properties; these include a renormalized energy spectrum with Kolmogorov's constant determined and an inhomogeneous ordinary differential equation for an invariant effective eddy viscosity, which allows for a closed-form solution that facilitates derivation of the Smagorinsky model for large-eddy simulation of turbulent flow.

In the study, we shall be concerned with turbulent transport properties for the passive scalar, in particular, the temperature field. There are several points of interest regarding these properties. First of all, estimation of the Batchelor constant has been an effort of research since Batchelor [2,3] proposed his scaling laws for the thermal turbulent energy spectrum respectively for $\mathrm{Pr}_0 \gg 1$ and $\mathrm{Pr}_0 \ll 1$. Many authors have made efforts in estimating or measuring the constant; different values were obtained, ranging from as low as 0.2 up to about 2 under various thermal flow conditions. The Prandtl number $\mathrm{Pr}_0 = \nu_0/\sigma_0$, defined to be the ratio of the molecular viscosity and thermal diffusivity and its turbulent analog Pr_t, could possibly be the single most important parameters in the study of thermal (turbulent) flow. Because of $\mathrm{Pr}_t = \nu_t/\sigma_t$, it is of great interest to obtain a ''not too complicated'' formula relating Pr_t to other characteristic flow numbers, e.g., the turbulent Peclet number $\mathrm{Pe}_t = \mathrm{Pr}_0 \nu_t/\nu_0$. We may establish such a relationship from which the thermal turbulent diffusivity σ_t can be easily read provided the viscous turbulent diffusivity ν_t has been obtained by other means. In a recent article, Kays [4] gave a convincing discussion of the role played by the turbulent Prandtl number Pr_t and had an excellent account of various results related to the number, both theoretical and experimental as well as numerical. There are several known results, in particular, from direct numerical simulation (DNS), about the turbulent Prandtl number Pr_t as a function of the turbulent Peclet number Pe_t; these include Kasagi *et al.* [5] for $\mathrm{Pr}_0 = 0.025$, and Bell *et al.* [6] as well as Kim and Moin [7] for $\mathrm{Pr}_0 = 0.1$. The DNS results were at present necessarily restricted to low Reynolds numbers: nonetheless, the well-documented data may serve as a useful check against theoretical as well as experimental results. There have been also numerous measured data from experiments that correlate the turbulent Prandtl number Pr_t to the turbulent Peclet number Pe_t under various thermal turbulent conditions. Of particular interest to the present study are Bremhost and Krebs [8], Sheriff and O'Kane [9], and Fuch [10]; these authors provided detailed measurements of Prandtl numbers in liquid sodium at the centerline of pipe flow by varying entry flow conditions.

The recursive renormalization group analysis is carried out for investigation of the points addressed above and other aspects as well. For this, we shall further take the hypothesis of quasinormal approximation [11] for the correlation between the velocity and temperature fields. In Sec. II, the basic idea of the recursive renormalization group analysis is briefly explained; we are then able to proceed with establishment of a recursive relationship for an effective thermal diffusivity σ_n between two successive steps of renormalization. In Sec. III, we derive the turbulent Prandtl number Pr_t as a function of the turbulent Peclet number Pe_t and give a simple relationship at large Peclet numbers for $\mathrm{Pr}_0 \ll 1$. The formula is comparable to the one derived by Yakhot *et al.* [12]; both theories give generally close agreement with the DNS results cited above. In particular, for $\mathrm{Pr}_0 = 0.01$, the present formula gives remarkable coincidence with the DNS results of Bell *et al.* [6] and Kim and Moin [7], while for $\mathrm{Pr}_0 = 0.025$ our results lie somewhat above those calculated by Kasagi *et al.* [5]. The relationship also shows close agreement with the measured data of Bremhost and Krebs [8] for $\mathrm{Pr}_0 = 0.0058$, and those adapted from Sheriff and O'Kane [9] and Fuch

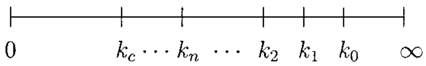

FIG. 1. The termini k_i for recursive renormalization with a fixed cutoff ratio $\Lambda = k_{n+1}/k_n$. Recursive renormalization analysis starts at the Kolmogorov's scale k_0 and ends at the cutoff wave number k_c.

[10] for $\mathrm{Pr}_0 \approx 0.0072$. In Sec. IV, an equation of an invariant effective thermal diffusivity σ as a function of the wave number k is sought with limiting operation of recursive renormalization. The equation turns out to be an inhomogeneous ordinary differential equation which is simple enough to render itself a closed-form solution. The invariant $\sigma(k)$, together with our previous result of the invariant effective viscous eddy viscosity $\nu(k)$, yields an invariant effective Prandtl number Pr as a function of the wave number k. The closed-form solution $\sigma(k)$, when combined with a modified Batchelor's thermal turbulent energy spectrum, further facilitates, respectively, in Secs. V and VI determination of the Batchelor constant C_B and derivation of a parallel Smagorinsky model for thermal turbulent transport and the model constant C_P; both constants C_B and C_P are found to depend upon several characteristic wave numbers. Finally, concluding remarks are drawn in Sec. VII.

II. RENORMALIZATION OF THE PASSIVE-SCALAR EQUATION

As in our previous study [1], the turbulence considered is isotropic, stationary in time and homogeneous in space. The flow is assumed to be governed by the incompressible Navier-Stokes equation, which in wave number space reads

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j\, u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t), \tag{1}$$

where

$$M_{\alpha\beta\gamma}(\mathbf{k}) = [k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})]/2i,$$

with

$$D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}.$$

The passive scalar, in particular the temperature field T, is governed by the thermal energy equation, which in wave number space reads

$$\left(\frac{\partial}{\partial t} + \sigma_0 k^2\right) T(\mathbf{k},t) = -ik_\alpha \int d^3j\, T(\mathbf{k}-\mathbf{j},t) u_\alpha(\mathbf{j},t). \tag{2}$$

The basic idea of recursive RG analysis is to divide the wave number space $(0,k_0)$, where k_0 is Kolmogorov's scale, to a supergrid region $(0,k_c)$ and a subgrid region (k_c,k_0); the subgrid modes are then removed shell by shell by taking the subgrid average over a spherical shell (k_{n+1},k_n), as shown in Fig. 1.

To distinguish the supergrid and subgrid modes, we introduce the commonly used notation as follows:

$$u_\alpha(\mathbf{k},t) = \begin{cases} u_\alpha^<(\mathbf{k},t) & \text{for } |\mathbf{k}|<k_1, \\ u_\alpha^>(\mathbf{k},t) & \text{for } |\mathbf{k}|>k_1, \end{cases}$$

and

$$T(\mathbf{k},t) = \begin{cases} T^<(\mathbf{k},t) & \text{for } |\mathbf{k}|<k_1, \\ T^>(\mathbf{k},t) & \text{for } |\mathbf{k}|>k_1. \end{cases}$$

Before substantial progress can be made with the renormalization group analysis, we shall take the following statistical hypotheses: (i) The turbulence fields have ensemble-mean-zero fluctuation:

$$\langle u_\alpha^>(\mathbf{k},t)\rangle = \langle T^>(\mathbf{k},t)\rangle = 0.$$

(ii) Supergrid components are considered to be statistically independent of subgrid averaging:

$$\langle u_\alpha^<(\mathbf{k},t)\rangle = u_\alpha^<(\mathbf{k},t), \quad \langle T^<(\mathbf{k},t)\rangle = T^<(\mathbf{k},t).$$

(iii) The triple moment of velocity and thermal passive scalar is considered to follow the quasi-normal approximation [11], as explained in Lesieur [13], p. 238:

$$\langle uuT\rangle = \langle uu\rangle\langle T\rangle.$$

Performing subgrid averaging of Eqs. (1) and (2) with the use of (i) and (ii), we may obtain, respectively, the equation for the supergrid velocity $u_\alpha^<$,

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^<(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j [u_\beta^<(\mathbf{j},t) u_\gamma^<(\mathbf{k}-\mathbf{j},t) \\ + \langle u_\beta^>(\mathbf{j},t) u_\gamma^>(\mathbf{k}-\mathbf{j},t)\rangle],\end{aligned} \tag{3}$$

and the equation for the supergrid temperature $T_\alpha^<$,

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \sigma_0 k^2\right) T^<(\mathbf{k},t) = -ik_\alpha \int d^3j [T^<(\mathbf{k}-\mathbf{j},t) u_\alpha^<(\mathbf{j},t) \\ + \langle T^>(\mathbf{k}-\mathbf{j},t) u_\alpha^>(\mathbf{j},t)\rangle].\end{aligned} \tag{4}$$

As for the subgrid components, we shall take the Markovian approximation by neglecting $\partial/\partial t$; this gives, respectively, the equation for the subgrid velocity $u_\alpha^>$,

$$\begin{aligned}(\nu_0 j^2) u_\alpha^>(\mathbf{j},t) = M_{\alpha\beta\gamma}(\mathbf{j}) \int d^3j' [u_\beta^<(\mathbf{j}',t) u_\gamma^<(\mathbf{j}-\mathbf{j}',t) \\ + 2u_\beta^<(\mathbf{j}',t) u_\gamma^>(\mathbf{j}-\mathbf{j}',t) \\ + u_\beta^>(\mathbf{j}',t) u_\gamma^>(\mathbf{j}-\mathbf{j}',t)],\end{aligned} \tag{5}$$

and the equation for the subgrid temperature $T_\alpha^>$,

$$(\sigma_0 k^2) T^>(\mathbf{k},t) = -ik_\alpha \int d^3j' [T^<(\mathbf{k}-\mathbf{j}',t) u_\alpha^<(\mathbf{j}',t) + T^>(\mathbf{k}-\mathbf{j}',t) u_\alpha^<(\mathbf{j}',t) + T^< \times (\mathbf{k}-\mathbf{j}',t) u_\alpha^>(\mathbf{j}',t) + T^>(\mathbf{k}-\mathbf{j}',t) \times u_\alpha^>(\mathbf{j}',t)]. \quad (6)$$

It was shown by our previous study that the second term on the right-hand side (RHS) of Eq. (3) has been ascribed to the production to the effective viscous eddy diffusivity. Parallel to this, we shall evaluate explicitly the second term on the RHS of Eq. (4), which contributes to the effective thermal eddy diffusivity.

For this purpose, we make use of Eq. (6) by renaming the index α by β and $\mathbf{k}$ by $\mathbf{k}-\mathbf{j}$, and then multiply on both sides by $u_\alpha^>(\mathbf{j},t)$, followed by subgrid ensemble averaging. This yields

$$\begin{aligned}(\sigma_0|\mathbf{k}-\mathbf{j}|^2)\langle T^>(\mathbf{k}-\mathbf{j},t) u_\alpha^>(\mathbf{j},t)\rangle &= -i(k_\beta - j_\beta)\int d^3j' [\langle T^>(\mathbf{k}-\mathbf{j}-\mathbf{j}',t) \\ &\quad\times u_\alpha^>(\mathbf{j},t) u_\beta^<(\mathbf{j}',t)\rangle \\ &\quad+ \langle T^<(\mathbf{k}-\mathbf{j}-\mathbf{j}',t) u_\alpha^>(\mathbf{j},t) u_\beta^>(\mathbf{j}',t)\rangle], \end{aligned} \quad (7)$$

where we have employed hypothesis (iii) to treat $\langle T^> \mathbf{u}^> \mathbf{u}^> \rangle$ as a zero subgrid-averaging term. The next step is to multiply $T^>(\mathbf{k}-\mathbf{j},t)$ on both sides of Eq. (5), and then take subgrid averaging to obtain

$$(\nu_0 j^2)\langle T^>(\mathbf{k}-\mathbf{j},t) u_\alpha^>(\mathbf{j},t)\rangle = 2M_{\alpha\beta\gamma}(\mathbf{j}) \int d^3j' \langle u_\beta^<(\mathbf{j}',t) u_\gamma^>(\mathbf{j}-\mathbf{j}',t) T^>(\mathbf{k}-\mathbf{j},t)\rangle. \quad (8)$$

Adding Eqs. (7) and (8) together gives

$$\begin{aligned}(\sigma_0|\mathbf{k}-\mathbf{j}|^2 + \nu_0 j^2)\langle T^>(\mathbf{k}-\mathbf{j},t) u_\alpha^>(\mathbf{j},t)\rangle &= \int d^3j' [-i(k_\beta - j_\beta)\langle T^>(\mathbf{k}-\mathbf{j}-\mathbf{j}',t) u_\alpha^>(\mathbf{j},t) \\ &\quad\times u_\beta^<(\mathbf{j}',t)\rangle + 2M_{\alpha\beta\gamma}(\mathbf{j})\langle u_\beta^<(\mathbf{j}',t) u_\gamma^>(\mathbf{j}-\mathbf{j}',t) \\ &\quad\times T^>(\mathbf{k}-\mathbf{j},t)\rangle - i(k_\beta - j_\beta)\langle T^<(\mathbf{k}-\mathbf{j}-\mathbf{j}',t) \\ &\quad\times u_\alpha^>(\mathbf{j},t) u_\beta^>(\mathbf{j}',t)\rangle]. \end{aligned} \quad (9)$$

According to hypothesis (iii), both the first and second terms on the RHS of the above equation vanish Eq. (9) simply reduces to

$$\begin{aligned}(\sigma_0|\mathbf{k}-\mathbf{j}|^2 + \nu_0 j^2)\langle T^>(\mathbf{k}-\mathbf{j},t) u_\alpha^>(\mathbf{j},t)\rangle &= -i(k_\beta - j_\beta)\int d^3j' \langle u_\alpha^>(\mathbf{j},t) u_\beta^>(\mathbf{j}',t)\rangle T^<(\mathbf{k}-\mathbf{j}-\mathbf{j}',t) \\ &= -i(k_\beta - j_\beta)\int d^3j' D_{\alpha\beta}(\mathbf{j})\,\delta(\mathbf{j}+\mathbf{j}')\,Q(\mathbf{j}) T^<(\mathbf{k}-\mathbf{j}-\mathbf{j}',t) \\ &= -ik_\beta D_{\alpha\beta}(\mathbf{j}) Q(\mathbf{j}) T^<(\mathbf{k},t), \end{aligned} \quad (10)$$

where we have used the velocity autocorrelation

$$\langle u_\alpha^>(\mathbf{j},t) u_\beta^>(\mathbf{j}',t)\rangle = D_{\alpha\beta}(\mathbf{j})\,\delta(\mathbf{j}+\mathbf{j}')\,Q(\mathbf{j})$$

and applied the obvious relation $j_\beta D_{\alpha\beta}(\mathbf{j}) = 0$ in the last equality.

Substituting Eq. (10) into Eq. (4), we finally obtain, after removing the first subgrid shell, the renormalized equation for the passive temperature T:

$$\left(\frac{\partial}{\partial t} + \sigma_1(k) k^2\right) T(\mathbf{k},t) = -ik_\alpha \int d^3j\; T(\mathbf{k}-\mathbf{j},t) u_\alpha(\mathbf{j},t) \quad (11)$$

for $0 < |\mathbf{k}| < k_1$ by introducing the effective thermal eddy diffusivity

$$\sigma_1(k) = \sigma_0 + \delta\sigma_0(k),$$

with the first increment given by

$$\begin{aligned}\delta\sigma_0(k) &= \frac{k_\alpha k_\beta}{4\pi k^2}\int_{\Omega_0} d^3j\, \frac{D_{\alpha\beta}(\mathbf{j}) E(\mathbf{j})}{[\sigma_0|\mathbf{k}-\mathbf{j}|^2 + \nu_0 j^2] j^2} \\ &= \frac{1}{4\pi}\int_{\Omega_0} d^3j\, \frac{(1-\mu^2) E(\mathbf{j})}{[\sigma_0|\mathbf{k}-\mathbf{j}|^2 + \nu_0 j^2] j^2}, \end{aligned}$$

where we have denoted by $E(j)$ the isotropic energy spectrum as $4\pi j^2 Q(\mathbf{j})$, and μ the direction cosine between the vectors $\mathbf{k}$ and $\mathbf{j}$, and thus $k_\alpha k_\beta D_{\alpha\beta}(\mathbf{j}) = k^2(1-\mu^2)$. The integration is over the set of intersection of two spheres:

$$\Omega_0(\mathbf{k}) = \{\mathbf{j} | k_1 < |\mathbf{j}|,\ \ |\mathbf{k}-\mathbf{j}| < k_0\}.$$

Repeating the renormalization procedure for removal of the nth subgrid shell yields the general recursive relationship for the effective thermal eddy diffusivity for $0 < |\mathbf{k}| < k_{n+1}$,

$$\sigma_{n+1}(k) = \sigma_n(k) + \delta\sigma_n(k), \quad (12)$$

with the nth increment given by

$$\delta\sigma_n(k) = \frac{1}{4\pi}\int_{\Omega_n} d^3j\, \frac{(1-\mu^2) E_n(\mathbf{j})}{[\sigma_n(k-j)|\mathbf{k}-\mathbf{j}|^2 + \nu_n(j) j^2] j^2}, \quad (13)$$

where $\nu_n(k)$ denotes the renormalized effective eddy viscosity and $E_n(j)$ the renormalized kinetic energy spectrum, both of which have been obtained by the present authors [1] in an early study. Again, the integration is over the set of intersection of two spheres:

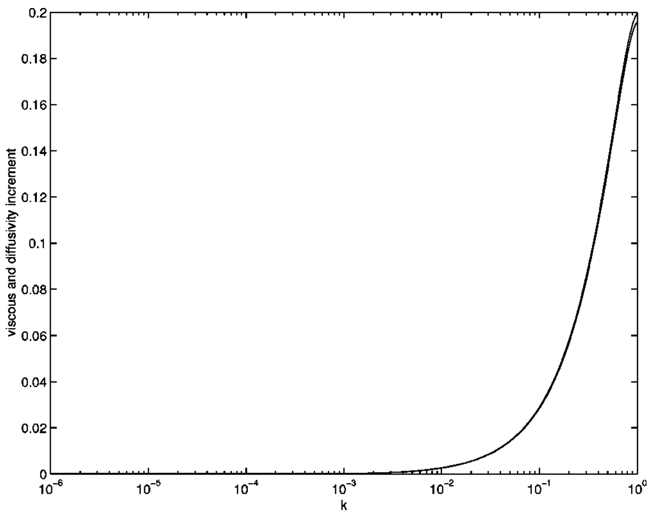

FIG. 2. The typical behavior of the increment of the effective eddy viscosity $\delta\nu_n$ and thermal eddy diffusivity $\delta\sigma_n$ versus the normalized wave number $k(0\leqslant k\leqslant 1)$, as the cutoff ratio Λ is close to 1.

$$\Omega_n(\mathbf{k})=\{\mathbf{j}|k_{n+1}<|\mathbf{j}|,\ \ |\mathbf{k}-\mathbf{j}|<k_n\}.$$

In parallel to Eq. (12), we have had for the effective momentum eddy viscosity

$$\nu_{n+1}(k)=\nu_n(k)+\delta\nu_n(k), \tag{14}$$

with the increment given by

$$\delta\nu_n(k)=\frac{1}{2\pi}\int_{\Omega_n}d^3j\,\frac{L(\mathbf{k},\mathbf{k}-\mathbf{j})E_n(\mathbf{j})}{j^2k^2[\nu_n(j)|\mathbf{j}|^2+\nu_n(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2]}, \tag{15}$$

where

$$L(\mathbf{k},\mathbf{k}-\mathbf{j})=\frac{(k^4-2k^3j\mu+kj^3\mu)(1-\mu^2)}{|\mathbf{k}-\mathbf{j}|^2}.$$

For later use, we record here

$$E_n(j)=A_pC_K\epsilon_n^{2/3}j^{-5/3}\exp\left(\frac{-3}{2}C_K^{-1/2}\epsilon_n^{-1/3}\nu_n(j)j^{4/3}\right), \tag{16}$$

where C_K is the Kolmogorov constant and ϵ_n is the rate of dissipation of the kinetic energy. The factor A_p is a modified function taking account of large-scale eddies given by

$$A_p(x)\equiv\frac{x^{s+5/3}}{1+x^{s+5/3}},$$

where $x=k/k_p$ with k_p denotes the peak of wave number of energy-containing eddies. As expected, the effective thermal eddy diffusivity $\sigma_n(k)$ for the passive scalar T should depend in some way on its counterpart $\nu_n(k)$ for the momentum; Eq. (13) exhibits explicitly such dependence. Figure 2 shows the typical trend, for the cutoff ratio Λ near unity, of the increment $\delta\sigma_n(k)$ of the effective thermal eddy diffusivity, which is shared with the increment $\delta\nu_n(k)$ of the effective eddy viscosity (or viscous eddy diffusivity). Both increments of viscous and thermal eddy diffusivities start with negligibly small values at lower wave numbers and rise rapidly near the cutoff wave number; this behavior is in good agreement with that obtained by Kraichnan [14] based on his testing field model (TFM).

III. TURBULENT PRANDTL NUMBER $\mathbf{Pr}_t$

The simplest approach for turbulent heat transfer is the Reynolds analog, i.e., $\nu_t=\sigma_t$ $(\mathrm{Pr}_t=1)$. However, as indicated by Kays [4], for most thermal boundary layers flows, the Reynolds analog is quite close to correct, but it still not precisely correct, and there are distinguished departures.

In this section, we will analyze, for each cutoff wave number k_c, the variation of the turbulent Prandtl number with the turbulent Peclet number. First of all, we evaluate the recursive relationship (12) for the effective thermal diffusivity at the terminal value k_{n+1}:

$$\sigma_{n+1}(k_{n+1})=\sigma_n(k_{n+1})+\delta\sigma_n(k_{n+1}).$$

By writing $\delta k=k_n-k_{n+1}$ and evaluating Eq. (13) at k_{n+1}, we readily obtain

$$\begin{aligned}\frac{\sigma_{n+1}(k_{n+1})-\sigma_n(k_{n+1})}{k_n-k_{n+1}}&=\frac{C_K\epsilon_n^{2/3}}{4\pi\delta k}\int_{\Omega_n(k_{n+1})}d^3j\\&\times\frac{(1-\mu^2)A_p(j/k_p)j^{-5/3}e^{-1.5C_K^{-1/2}\epsilon_n^{-1/3}\nu_n(j)j^{4/3}}}{[\sigma_n(k-j)|\mathbf{k}-\mathbf{j}|^2+\nu_n(j)j^2]j^2},\end{aligned} \tag{17}$$

where $A_p=1$ as $j\gg k_p$. In the limit of $n\gg 1$, the integral on the RHS is of order δk and the numerator of the LHS of the above equation must be in magnitude of the order δk. By taking the limit, we have $\delta k\ll 1$, $k_n\to k_c$, $\nu_n\to\nu$, $\sigma_n\to\sigma$, and the measure of $\Omega_n(k_{n+1})$ is of the order δk, where $\nu=\nu(k_c)$ and $\sigma=\sigma(k_c)$ denote, respectively, the invariant effective viscous and thermal eddy diffusivities. The measure of the set $\Omega_n(k_{n+1})$ given below Eq. (13), in the limit of $k_{n+1}\to k_c$, can be approximated by

$$\Pi(\Omega(k_{n+1}))=2\pi\sqrt{1-\left(\frac{1}{2}\right)^2}\frac{k_c^3}{\sqrt{1-(\frac{1}{2})^2}}\frac{\delta k}{k_c}+O(\delta k^2).$$

It follows from these facts that Eq. (17) is led to the differential version

$$\frac{d\sigma(k_c)}{dk_c} \approx \frac{C_K \xi^{2/3}}{4\pi\delta k}\int_{\Omega(k_c)} d^3 j \times \frac{[1-(k/2j)^2]j^{-5/3}e^{-1.5C_K^{-1/2}\epsilon^{-1/3}\nu(j)j^{1/3}}}{[\sigma(k-j)|\mathbf{k}-\mathbf{j}|^2+\nu(j)j^2]j^2}$$
$$\rightarrow \frac{C_K \epsilon^{2/3} e^{-1.5B(1)k_c^{-11/3}[1-(\frac{1}{2})^2]}}{2[\sigma(k_c)+\nu(k_c)]}, \tag{18}$$

where $B(1)=C_K^{-1/2}\epsilon^{-1/3}k_c^{4/3}\nu(k_c)$ and has been estimated to be 0.6633 from our previous analysis [1]. The same argument applies to the effective viscous eddy diffusivity and leads to the differential expression

$$\frac{d\nu(k_c)}{dk_c} = \frac{C_K \epsilon^{2/3} e^{-1.5B(1)K_c^{-11/3}[1-(\frac{1}{2})^2]}}{4\nu(k_c)}. \tag{19}$$

Equipped with the results of Eqs. (18) and (19), we are now able to proceed with derivation of an algebraic equation relating the turbulent Prandtl number Pr_t to the turbulent Peclet number Pe_t. For convenience of calculation, we define α to be the inverse of the effective eddy Prandtl number $\mathrm{Pr}(k_c)$ and write

$$\alpha(k_c) \equiv \mathrm{Pr}^{-1}(k_c) \equiv \frac{\sigma(k_c)}{\nu(k_c)} = \frac{\sigma_0+\sigma_t(k_c)}{\nu_0+\nu_t(k_c)}, \tag{20}$$

where σ_0 and ν_0 denote in turn the molecular thermal and viscous diffusivities, while ν_t and σ_t are, respectively, called the thermal and viscous turbulent eddy diffusivities. The Prandtl number is defined to be $\mathrm{Pr}_0=\nu_0/\sigma_0$; we may further define the turbulent Prandtl number $\mathrm{Pr}_t=\nu_t/\sigma_t$ and also the turbulent Peclet number $\mathrm{Pe}_t=\mathrm{Pr}_0\nu_t/\nu_0$. Then α can further be rewritten as

$$\alpha = \frac{1+\mathrm{Pe}_t\mathrm{Pr}_t^{-1}}{\mathrm{Pr}_0+\mathrm{Pe}_t}. \tag{21}$$

Differentiating α with respect to k_c in Eq. (20) with use of Eqs. (18) and (19) yields

$$\frac{d\alpha(k_c)}{dk_c} = \left\{\frac{C_K \epsilon^{2/3} e^{-1.5B(1)k_c^{-11/3}[1-(\frac{1}{2})^2]}}{2[\sigma(k_c)+\nu(k_c)]}\right\}\frac{1}{\nu(k_c)} - \alpha\frac{\nu'(k_c)}{\nu(k_c)}$$
$$= \left(\frac{2}{\alpha+1}-\alpha\right)\frac{\nu'(k_c)}{\nu(k_c)} \quad [\text{since Eq. (14)}]. \tag{22}$$

Equation (22) can be easily integrated to give

$$\ln[(\alpha+2)^{-1/3}(\alpha-1)^{-2/3}] = \ln\nu(k_c)+C,$$

where the constant C should be chosen such that $\nu(k_c)\rightarrow\nu_0$ and $\sigma(k_c)\rightarrow\sigma_0$ as $k_c\rightarrow k_0$ (with k_0 denoting the Kolmogorov scale), and thus

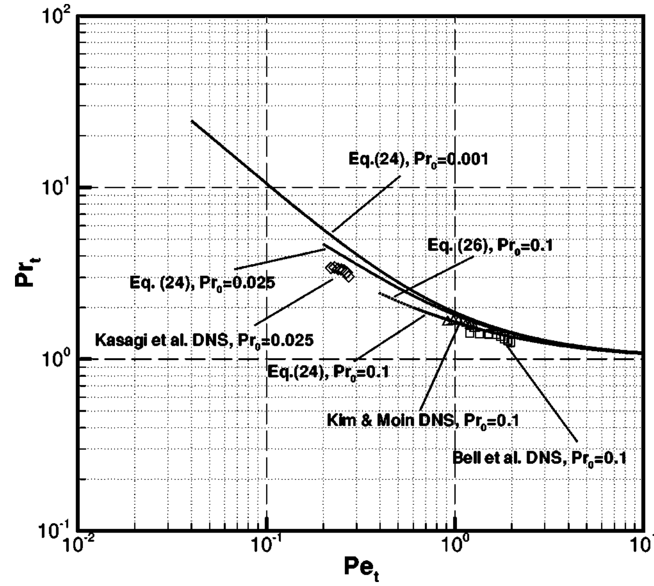

FIG. 3. The turbulent Prandtl number Pr_t versus the turbulent Peclet number Pe_t, plotted based on Eqs. (24) and (21) for $\mathrm{Pr}_0=0.001$, 0.025, and 0.1 with comparisons to numerical results from DNS by several authors, as shown.

$$\left(\frac{\alpha+2}{\alpha_0+2}\right)^{1/3}\left(\frac{\alpha-1}{\alpha_0-1}\right)^{2/3} = \frac{\nu_0}{\nu(k_c)}. \tag{23}$$

Recalling $\nu(k_c)=\nu_0+\nu_t(k_c)$, we may rewrite the RHS of the above equation to obtain

$$\left(\frac{\alpha+2}{\alpha_0+2}\right)^{1/3}\left(\frac{\alpha-1}{\alpha_0-1}\right)^{2/3} = \frac{1}{1+\mathrm{Pe}_t/\mathrm{Pr}_0}. \tag{24}$$

It is of great interest to compare Eq. (24) with a similar formula obtained by Yakhot *et al.* [12], which reads, in terms of the present notation,

$$\left(\frac{\alpha+2.1793}{\alpha_0+2.1793}\right)^{0.35}\left(\frac{\alpha-1.1793}{\alpha_0-1.1793}\right)^{0.65} = \frac{1}{1+\mathrm{Pe}_t/\mathrm{Pr}_0}. \tag{25}$$

The apparent resemblance between the formulas (24) and (25) is quite remarkable not only because of the similarity in mathematical structure, but because the results are basically derived entirely from different approaches to renormalization group analysis of turbulence. According to Eqs. (24) and (21), Pr_t can be represented in terms of Pe_t for any given Prandtl number Pr_0. Figure 3 shows the results for three different values of $\mathrm{Pr}_0=0.01$, 0.025 and 0.1. Comparison with the DNS (direct-numerical-simulation) results of Bell *et al.* [6] and Kim and Moin [7] shows excellent agreement for $\mathrm{Pr}_0=0.1$, while there is a mild discrepancy with that of Kasagi *et al.* [5]. For comparison, we recall from Kay [4] that it is at $\mathrm{Pr}_0=0.01$ rather than $\mathrm{Pr}_0=0.1$ that the result based on the formula of Yakhot *et al.* showed good agreement with the cited DNS results, which were all obtained at $\mathrm{Pr}_0=0.1$. For comparison with experiments, Fig. 4 shows also close agreement of the present formula for $\mathrm{Pr}_0=0.0058$ (which has little difference while for $\mathrm{Pr}_0=0.0072$) with the

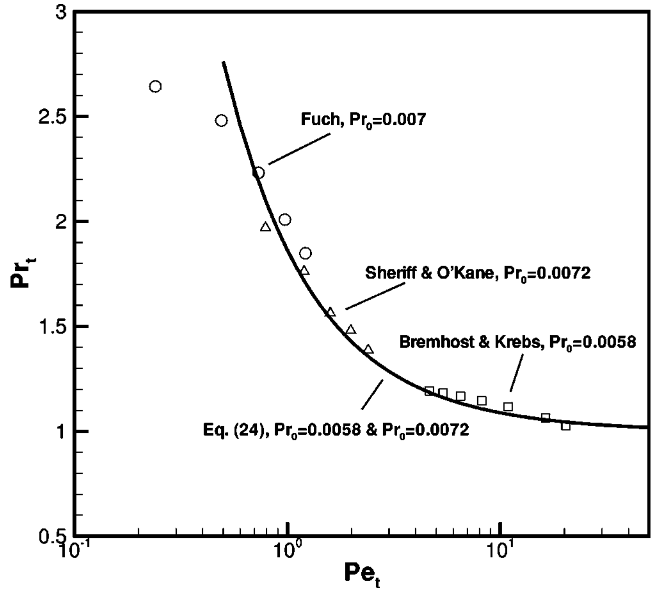

FIG. 4. The turbulent Prandtl number Pr_t versus the turbulent Peclet number Pe_t plotted based on Eqs. (24) and (21) for $\mathrm{Pr}_0=0.058$ (or 0.072) along with the experimental data measured by Bremhost and Krebs [8], Sheriff and O'Kane [9], and Fuch [10].

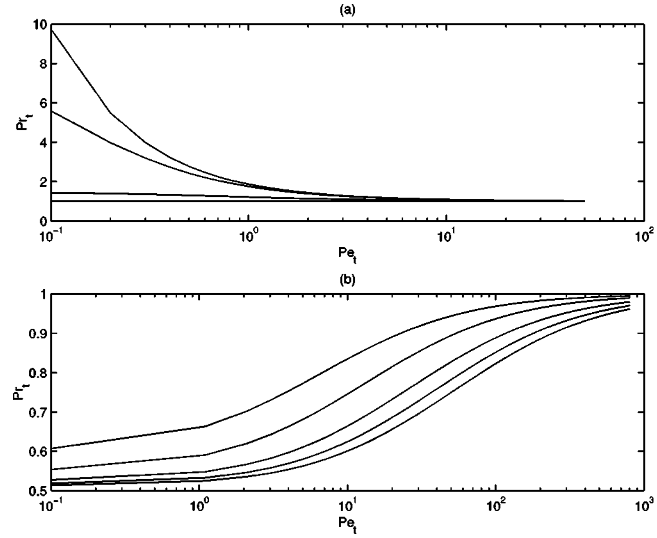

FIG. 5. Curves of the turbulent Prandtl number Pr_t versus the turbulent Peclet number Pe_t drawn based on Eqs. (24) and (21). Part (a) for $\mathrm{Pr}_0\leqslant 1$, which is from above 0.005, 0.05, 0.5, and 1; part (b) for $\mathrm{Pr}_0>1$, which is from above 5, 10, 20, 30, and 40.

experimental data taken from Bremhost and Krebs [8] for $\mathrm{Pr}_0=0.0058$, Sheriff and O'Kane [9] for $\mathrm{Pr}_0=0.0072$, and Fuch [10] for $\mathrm{Pr}_0=0.007$, all measured in liquid sodium at the centerline of thermal turbulent pipe flow. The latter two references did not report Pr_t directly in terms of Pe_t; the data shown are adapted from the references by appropriate scalings; Fuch's result, however, still deviates from the present formula at small Pe_t.

The formula determined by Eq. (24) with α given by Eq. (21) is of general interest in several aspects. For each α_0 (or Pr_0), Eq. (24) is an algebraic equation of degree 3; there is always a solution to it. For the particular $\mathrm{Pr}_0=1$, it is noticed that the Reynolds analog, i.e., $\mathrm{Pr}_t=1$, always holds true no matter what the turbulent Peclet number Pe_t is, but for $\mathrm{Pr}_0\neq 1$, the Reynolds analogy can only hold true in the limit of $\mathrm{Pe}_t\gg 1$. Figures 5(a) and 5(b) show, respectively, a few curves of Pr_t versus Pe_t for $\mathrm{Pr}_0\leqslant 1$ and for $\mathrm{Pr}_0>1$. For $\mathrm{Pr}_0<1$, the turbulent Prandtl number Pr_t is observed to decrease monotonically to 1 as Pe_t is increased indefinitely, while for $\mathrm{Pr}_0>1$, Pr_t increases with increasing Pe_t, and finally saturates to 1. For either $\mathrm{Pr}_0<1$ or $\mathrm{Pr}_0>1$, the curve of Pr_t versus Pe_t is higher for larger Pr_0; that is, for given Pe_t, the magnitude of Pr_t would decrease as Pr_0 is increased. For each $\mathrm{Pr}_0<1$, there is a concise alternative which shows excellent approximation to the general formula, Eq. (24), and which is given by

$$\mathrm{Pe}_t=\frac{H}{\mathrm{Pe}_t^{\theta}}+1, \tag{26}$$

where H and θ are positive integers that depend upon Pr_0. An example of this is the case of $\mathrm{Pr}_0=0.1$, for which we have good approximation by setting $H=0.64$ and $\theta=0.87$ in Eq. (26). It is also of interest to note that a similar empirical formula based on experimental data has been proposed by Bremhost and Krebs [8]. Finally, we noticed that the Peclet number Pe_t is a thermal analog of the Reynolds number. The reason why Pr_t always tends to 1 as Pe_t becomes sufficiently large would be that at this time the thermal flow turbulence is completely convection dominated and the temperature field T responds immediately to any change of the velocity field $\mathbf{v}$.

IV. INVARIANT EFFECTIVE THERMAL EDDY DIFFUSIVITY

The purpose of the section is to determine the limiting solution to the recursive renormalization equation (12) for the effective thermal eddy diffusivity σ_n, as the cutoff ratio Λ tends to 1.

First of all, we rescale the wave number k by setting $\tilde{k}=k/k_{n+1}$. Suggested by our previous scaling result for the effective eddy viscosity $\nu_n(k)=\tilde{\nu}_n(\tilde{k})k_n^{-4/3}$, we propose also a similar scaling law for the effective thermal eddy diffusivity $\sigma_n(k)=\tilde{\sigma}_n(\tilde{k})k_n^{-4/3}$ and rewrite Eq. (13) in the form

$$\delta\sigma_n(k)=\frac{C_K\epsilon_n^{2/3}k_{n+1}^{-4/3}}{4\pi}\int_{\tilde{\Omega}_n}d^3\tilde{j}\times\frac{(1-\mu^2)A_p\tilde{j}^{-5/3}\exp(-\frac{3}{2}C_K^{-1/2}\tilde{\nu}_n(\tilde{j})\epsilon_n^{-1/3}\tilde{j}^{4/3}}{[\tilde{\sigma}_n(\tilde{k}-\tilde{j})|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|^2+\tilde{\nu}_n(\tilde{j})\tilde{j}^{\,2}]\tilde{j}^{\,2}}. \tag{27}$$

The recursive equation (12) can then be recast in the rescaled dimensionless form

$$\tilde{\sigma}_{n+1}(\tilde{k})-\Lambda^{-4/3}\tilde{\sigma}_n(\tilde{k}\Lambda)=\Lambda^{-4/3}\delta\tilde{\sigma}_n(\tilde{k}\Lambda). \tag{28}$$

Now we write $\Lambda = 1-\xi$, and let $n\to\infty$. Equivalently, we have $\xi\to 0$, $\tilde{\sigma}_n\to\tilde{\sigma}$, and thus Eq. (28) becomes for $n\gg 1$

$$\begin{aligned}\left[\tilde{k}\frac{d\tilde{\sigma}(\tilde{k})}{d\tilde{k}}+\frac{4}{3}\tilde{\sigma}(\tilde{k})\right]\xi &= \frac{(1-\xi)^{-4/3}}{4\pi}C_K\epsilon^{2/3}\int_{\tilde{\Omega}}d^3\tilde{j}\\ &\quad\times\frac{(1-\mu^2)A_p\tilde{j}^{-5/3}\exp(-\frac{3}{2}C_K^{-1/2}\tilde{\nu}\epsilon^{-1/3}\tilde{j}^{4/3})}{\tilde{j}^2[\tilde{\nu}(\tilde{j})|\tilde{\mathbf{j}}|^2+\tilde{\sigma}(|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|)|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|^2]}\\ &= \frac{C_K\epsilon^{2/3}}{4\pi}\frac{A_p^c\exp(-1.5C_K^{-1/2}\tilde{\nu}(1)\epsilon^{-1/3})}{\tilde{\nu}(1)+\tilde{\sigma}(1)}\\ &\quad\times\left[1-\left(\frac{\tilde{k}}{2}\right)^2\right]\Pi(\tilde{\Omega})+O(\xi^2), \end{aligned}\tag{29}$$

where $A_p^c = A_p(k_c/k_p)$ and we have known $\Pi(\tilde{\Omega}_n)=2\pi\tilde{k}\xi + O(\xi^2)$, as $n\gg 1$. And therefore Eq. (29) in the limit of $\xi\to 0$ becomes

$$\begin{aligned}\tilde{k}\frac{d\tilde{\sigma}(\tilde{k})}{d\tilde{k}}+\frac{4}{3}\tilde{\sigma}(\tilde{k}) &= \frac{C_K\epsilon^{2/3}}{2}\frac{A_p^c\exp(-1.5C_K^{-1/2}\tilde{\nu}(1)\epsilon^{-1/3})}{\tilde{\nu}(1)+\tilde{\sigma}(1)}\\ &\quad\times\tilde{k}\left[1-\left(\frac{\tilde{k}}{2}\right)^2\right].\end{aligned}\tag{30}$$

This is an inhomogeneous ordinary differential equation that we seek for the rescaled invariant effective thermal eddy diffusivity $\tilde{\sigma}(\tilde{k})$. The equation is simple enough to render itself a closed-form solution. To simplify the expression, let us introduce $B_T(\tilde{k}) = C_K^{-1/2}\epsilon^{-1/3}\tilde{\sigma}(\tilde{k})$, and rewrite Eq. (30) in a more compact form

$$\tilde{k}\frac{dB_T(\tilde{k})}{d\tilde{k}}+\frac{4}{3}B_T(\tilde{k})=\frac{\exp[-1.5B(1)]}{2[B_T(1)+B(1)]}\tilde{k}\left[1-\left(\frac{\tilde{k}}{2}\right)^2\right].\tag{31}$$

Equation (31) can be easily integrated to give the exact solution

$$B_T(\tilde{k})=\left(B_T(1)-\frac{135}{364}\tilde{C}_\sigma\right)\tilde{k}^{-4/3}+\tilde{C}_\sigma\left(\frac{3}{7}\tilde{k}-\frac{3}{52}\tilde{k}^3\right),\tag{32}$$

where $B_T(1)$ is to be determined below and $\tilde{C}_\sigma$ is given by

$$\tilde{C}_\sigma = A_p^c\exp[-1.5B(1)]/\{2[B_T(1)+B(1)]\}.$$

If we return $B_T(\tilde{k})$ to $\sigma(k)$, the leading term of $\sigma(k)$ is simply proportional to $\epsilon^{1/3}k^{-4/3}$; this scaling result is exactly Richardson's four-thirds law (cf. [15], p. 556). To determine $B_T(1)$, we have had the estimate $B(1)=0.6633$, and thus $B_T(1)$ can be found from the turbulent Prandtl number $\mathrm{Pr}_t = 1$ in the limit of large Pe_t [cf. Eq. (24)]. It follows from Eq. (21) that $\mathrm{Pe}_t\gg 1$ for moderate Pr_0 implies $\alpha = 1/\mathrm{Pr}_t = 1$; then, we have

$$1=\alpha(k_c)=\frac{\sigma(k_c)}{\nu(k_c)}=\frac{C_K^{1/2}\epsilon^{1/3}k_c^{-4/3}B_T(1)}{C_K^{1/2}\epsilon^{1/3}k_c^{-4/3}B(1)}=\frac{B_T(1)}{0.6633}$$

and therefore $B_T(1)=B(1)=0.6633$. Substituting the value in the expression of $\tilde{C}_\sigma$, we finally have the same form as $B(\tilde{k})$ for the solution $B_T(\tilde{k})$:

$$B_T(\tilde{k})=B(\tilde{k})=0.6116\tilde{k}^{-4/3}+0.1394\left(\frac{3}{7}\tilde{k}-\frac{3}{52}\tilde{k}^3\right).\tag{33}$$

V. EVALUATION OF THE BATCHELOR CONSTANT

Batchelor [2,3] was the first to propose scaling laws for the thermal energy spectrum respectively for $\mathrm{Pr}_0\gg 1$ and $\mathrm{Pr}_0\ll 1$, leaving a proportional constant C_B undetermined. Subsequently, many authors have tried to estimate or measure the value of C_B under various flow conditions.

In theory, Kraichnan [14] had the estimate $C_B=0.208$, and Gibson [16] had the estimate $C_B=0.9$, while Yakhot and Orszag [17] obtained $C_B=1.16$ according to their ϵ-based renormalization group analysis. Kerr [18] had the numerical value $C_B=0.6$. In experiment, Gibson and Schwarz [19] had $C_B=0.35$, Grant *et al.* [20] obtained $C_B=0.31$, and Boston and Burling [21] had $C_B=0.81$, while Gurvich and Zubkovski [22] obtained $C_B=2.0$.

In the present study, we shall show that Batchelor's constant actually depends on Pr_0 as well as some characteristic wave numbers to be described below. First of all, let us modify the original Batchelor's thermal energy spectrum by introducing two connecting functions B_1 and B_2, respectively, for $\mathrm{Pr}_0\ll 1$ and $\mathrm{Pr}_0\gg 1$ so that the scaling laws with an appropriate choice of the proportional constant may fit well the experimental data. Let χ denote the dissipation rate of the thermal energy; we have for $\mathrm{Pr}_0\ll 1$,

$$E_T(k)=C_BB_1(x)\chi\epsilon^{-1/3}k^{-5/3},\tag{34}$$

where

$$B_1(x)\equiv\frac{x^{-4}}{1+x^{-4}}\quad\text{with}\quad x=\frac{k}{k_{cd}};$$

for $\mathrm{Pr}_0\gg 1$,

$$E_T(k)=C_BB_2(x)\chi\epsilon^{-1/3}k^{-5/3}e^{-2\sigma k^2\sqrt{\nu/\epsilon}},\tag{35}$$

where

$$B_2(x)\equiv\frac{x^{2/3}(1+x^{-1})}{1+x^{-1/3}}\quad\text{with}\quad x=\frac{k}{k_{ds}}.$$

In these expressions, k_{cd} is the conductive wave number which is of the order $(\epsilon/\sigma^3)^{1/4}$, while k_{ds} is the dissipation

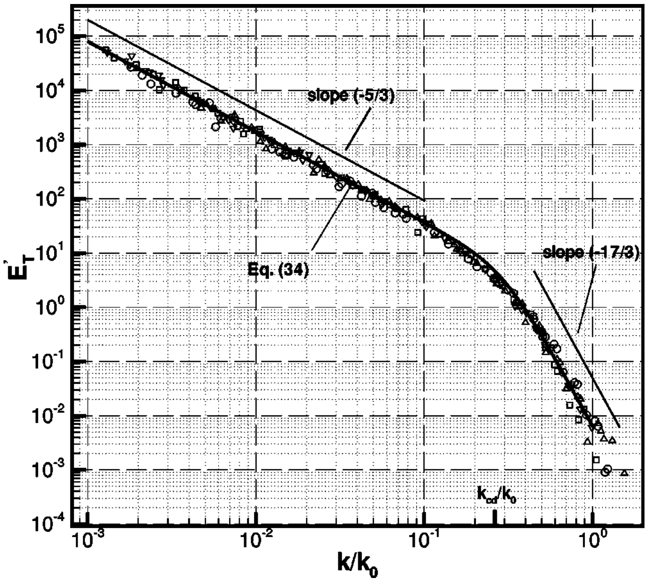

FIG. 6. The scaling law of the thermal energy spectrum $E_T(k)$ versus the normalized wave number k/k_0 for $\mathrm{Pr}_0 \ll 1$, defined by Eq. (34) with a proper choice of the proportional constant, fits well the measured data taken from Boston and Burling [21].

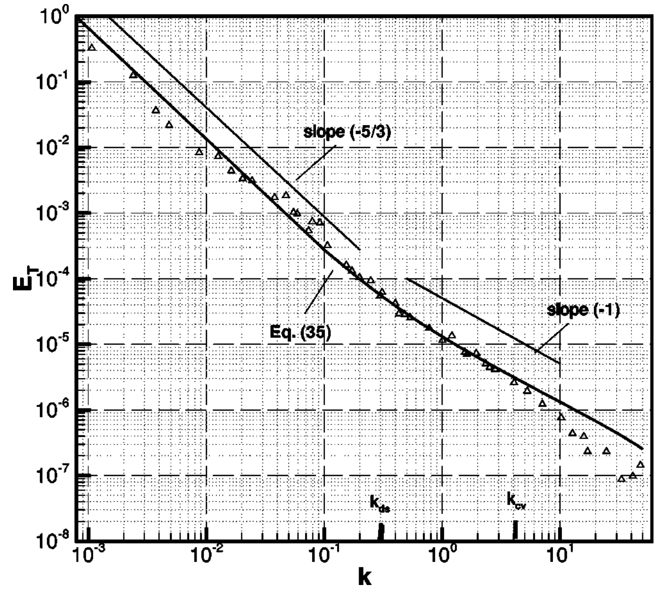

FIG. 7. The plot shows that the scaling law of the thermal energy spectrum $E_T(k)$ versus the wave number k for $\mathrm{Pr}_0 \gg 1$, defined by Eq. (35) with a proper choice of the proportional constant, fits well the measured data taken from Grant *et al.* [20]

wave number of the order $(\epsilon/\nu^3)^{1/4}$. There are two other important characteristic wave numbers k_s and k_p, which denote, respectively, the wave number of the largest eddy existing in the flow and the peak wave number of the energy-containing eddies.

Figure 6 shows that the measured data of Boston and Burling [21] fit excellently with the curve plotted based on Eq. (34) for $\mathrm{Pr}_0 \ll 1$ with the value of k_{cd} as indicated. Boston and Burling made measurements in air at a height of $4m$ over a tidal mud flat; their data showed that k_p/k_0 and k_s/k_0 are almost identical and take the value about 0.001, while k_{cd}/k_0 locates at about 0.03. Air-H_2O at 290 K has Prandtl number $\mathrm{Pr}_0 \approx 0.73$. From the figure, we observe that Eq. (34) follows a power law of $-5/3$ in the inertial-convective range of $k_s < k < k_{cd}$ and a power law of $-17/3$ in the inertial-conductive range of $k_{cd} < k < k_0$.

Figure 7 shows that the measured data of Grant *et al.* [20] is in good agreement with the curve plotted based on Eq. (35) for $\mathrm{Pr}_0 \gg 1$ with the value of k_{ds} as indicated. Grant *et al.* performed experiments in the open sea and in a tidal channel; their data showed that k_0 is about the order of 40, k_{vc} is of the order 1, and k_s/k_0 and k_p/k_0 are almost identical and take the value about 0.001, while k_{ds}/k_0 locates at about 0.06. Liquid water at 293 K has Prandtl number $\mathrm{Pr}_0 \approx 6.99$. From the figure, we observe that Eq. (35) followed a power law of $-5/3$ in the inertial-convective range of $k_s < k < k_{ds}$ and a power law of -1 in the viscous-convective range of $k_{ds} < k < k_{vc}$, where k_{vc} is of the order $(\epsilon/\sigma^2\nu)^{1/4}$.

With Eqs. (34) and (35), we are now able to determine the Batchelor constant; it is natural to require

$$\chi = \int_{k_s}^{k_c} 2\sigma(k)k^2 E_T(k)\,dk = \begin{cases} \displaystyle\int_{k_s}^{k_c} 2C_K^{1/2} C_B \chi k^{1/3} F_T(k) B_1(k/k_{cd})\,dk & (\text{for } \mathrm{Pr}_0 \ll 1), \\ \displaystyle\int_{k_s}^{k_c} 2C_K^{1/2} C_B \chi k^{1/3} F_T(k) B_2(k/k_{ds}) e^{-2\sigma k^2 \sqrt{\nu/\epsilon}}\,dk & (\text{for } \mathrm{Pr}_0 \gg 1) \end{cases}, \tag{36}$$

where the function F_T is introduced by setting $\sigma(k) = C_K^{1/2}\epsilon^{1/3}F_T(k)$ in analogy with its viscous counterpart $\nu(k) = C_K^{1/2}\epsilon^{1/3}F(k)$, and thus we have $F_T(k) = B_T(k)k_c^{-4/3}[F(k) = B(k)k_c^{-4/3}]$. Rearrangement of Eq. (36) gives

$$C_B = \begin{cases} \left[2C_K^{1/2}\displaystyle\int_{k_s}^{k_c} k^{1/3} F_T(k) B_1(k/k_{cd})\,dk\right]^{-1} & (\text{for } \mathrm{Pr}_0 \ll 1), \\ \left[2C_K^{1/2}\displaystyle\int_{k_s}^{k_c} k^{1/3} F_T(k) B_2(k/k_{ds}) e^{-2C_K^{3/4}F_T\sqrt{F}k^2}\,dk\right]^{-1} & (\text{for } \mathrm{Pr}_0 \gg 1). \end{cases} \tag{37}$$

Figure 8(a) shows a plot of C_B versus k_c/k_0 based on Eq. (37) for $\mathrm{Pr}_0 \ll 1$, from which it is clear to see also the dependence of C_B on the characteristic wave numbers: the conductive wave number k_{cd} and the peak energy-containing wave number k_p (or k_s). There are two trends of interest to be noted: (i) The Batchelor constant C_B increases with increasing the normalized k_c/k_0, and (ii) C_B increases as the conductive ratio k_{cd}/k_0 is increased. The plot of this part varies in a range of characteristic wave numbers of thermal energy spectrum that include those measured by Boston and Burling [21] whose experiment gave $C_B = 0.81$, which is, however, substantially higher than the values presented in the figure. Figure 8(b) is a plot of C_B versus k_c/k_0 based on Eq. (37) for $\mathrm{Pr}_0 \gg 1$, from which it is clear to see also the dependence of C_B on the characteristic wave numbers: the dissipative wave number k_{ds} and the peak energy-containing wave number k_p (or k_s). There are also two trends of interest to be noted: (i) The Batchelor constant C_B decreases with increasing the normalized k_c/k_0, and (ii) C_B increases as the dissipative ratio k_{ds}/k_0 is increased. The plot of this part varies in a range of characteristic wave numbers that include those measured by Grant *et al.* [20] whose experiment gave $C_B = 0.31$ in good agreement with the values presented in the figure. These observations indicate clearly that C_B is not a universal constant, but depends on the shape of the actual thermal energy spectrum $E_T(k)$.

Equation (37) is in integral form, and the integration may be carried out if we make the approximation by taking only the leading terms of $F(k)$ and $F_T(k)$, that is, $F_T(k) = F_T \simeq 0.6116 k^{-4/3}$; this yields the more explicit formula

$$C_B = \begin{cases} \left\{ 1.3942 C_K^{1/2} \left[\frac{1}{4} \ln \frac{(k_s/k_{cd})^4 + 1}{(k_c/k_{cd})^4 + 1} + \ln \frac{k_c}{k_s} \right] \right\}^{-1} & (\text{for } \mathrm{Pr}_0 \ll 1), \\ \left\{ 1.3942 C_K^{1/2} e^{-1.09 C_K^{3/4}} \left(\ln \frac{k_c}{k_s} + \frac{3}{2} R_1 - 3 R_2 \right) \right\}^{-1} & (\text{for } \mathrm{Pr}_0 \gg 1), \end{cases} \tag{38}$$

where

$$R_1 = \left(\frac{k_c}{k_{ds}} \right)^{2/3} - \left(\frac{k_s}{k_{ds}} \right)^{2/3}$$

and

$$R_2 = \left(\frac{k_c}{k_{ds}} \right)^{1/3} - \left(\frac{k_s}{k_{ds}} \right)^{1/3}.$$

VI. SMAGORINSKY MODEL FOR A PASSIVE SCALAR

The usual Smagorinsky model is used for the large-eddy simulation of Navier-Stokes equation. In this section, we will construct a parallel model for turbulent transport of the passive scalar. First of all, we express the rate of dissipation of the kinetic energy ϵ in the resolvable velocity:

$$\epsilon = \frac{\nu(k_c)}{2} \left(\frac{\partial \mathbf{u}_i^<}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j^<}{\partial \mathbf{x}_i} \right)^2. \tag{39}$$

Recall $\sigma(k) = C_K^{1/2} \epsilon^{1/3} B_T(\tilde{k}) k_c^{-4/3}$ and evaluate Eq. (32) at k_c to obtain

$$\sigma(k_c) = C_K^{1/2} \epsilon^{1/3} B_T(1) k_c^{-4/3}.$$

Replacing ϵ on the RHS of the above equation by Eq. (39) enables us to obtain

$$\sigma(k_c) = 0.6633 C_K^{1/2} \left[\frac{\nu(k_c)}{2} \left(\frac{\partial \mathbf{u}_i^<}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j^<}{\partial \mathbf{x}_i} \right)^2 \right]^{1/3} k_c^{-4/3}. \tag{40}$$

Specifically, let us consider turbulent flow with $\mathrm{Pe}_t \gg 1$, for which we have $\mathrm{Pr}(k_c) \approx \mathrm{Pr}_t(k_c)$ according to Eq. (21). Solving then the algebraic equation (40) for $\sigma(k_c)$ with k_c replaced by $2\pi/\Delta$ where Δ denotes the cutoff wavelength gives

$$\begin{aligned} \sigma(k_c) &= \frac{0.6633^{3/2}}{4\sqrt{2}\pi^2} \sqrt{\mathrm{Pr}_t(k_c)} C_K^{3/4} \Delta^2 \left| \frac{\partial \mathbf{u}_i^<}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j^<}{\partial \mathbf{x}_i} \right| \\ &\equiv C_P(k_c) \Delta^2 \left| \frac{\partial \mathbf{u}_i^<}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j^<}{\partial \mathbf{x}_i} \right|. \end{aligned} \tag{41}$$

Equation (41) is the thermal Smagorinsky model for the passive scalar T, and the model constant C_P is given by

$$C_P(k_c) = \frac{0.6633^{3/2}}{4\sqrt{2}\pi^2} \sqrt{\mathrm{Pr}_t(k_c)} C_K^{3/4}(k_c) = 0.0097 C_K^{3/4}(k_c),$$

where we may simply take $\mathrm{Pr}_t(k_c) = 1$, implied by $\mathrm{Pe}_t \gg 1$ [cf. Eq. (24)]. Figure 9 shows how the Smagorinsky model constant C_P varies with k_c/k_0. As the normalized k_c/k_0 moves away from 1, the model constant C_P is increasing, and the constant increases also with increasing the normalized peak wave number k_p/k_0 of energy-containing eddies.

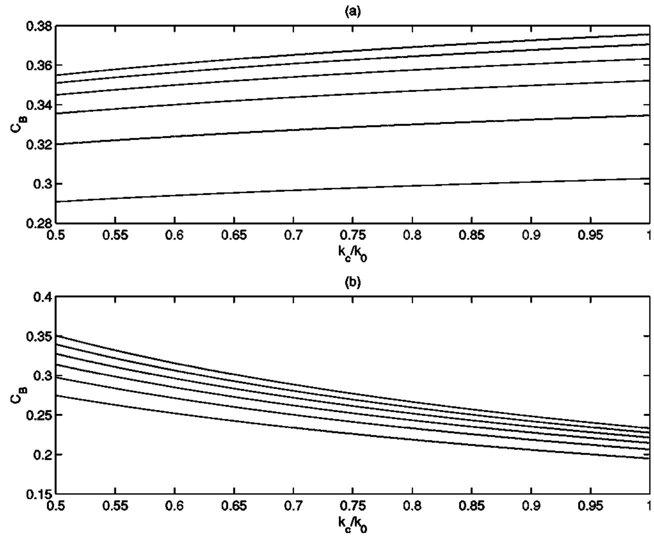

FIG. 8. The behavior of C_B vs k_c/k_0 for the flow with $0.001 < k_s/k_0 < 0.006$ and $0.002 < k_p/k_0 < 0.012$. Part (a) for $\mathrm{Pr}_0 \ll 1$ with $0.05 < k_{cd}/k_0 < 0.1$; part (b) for $\mathrm{Pr}_0 \gg 1$ with $0.05 < k_{ds}/k_0 < 0.055$.

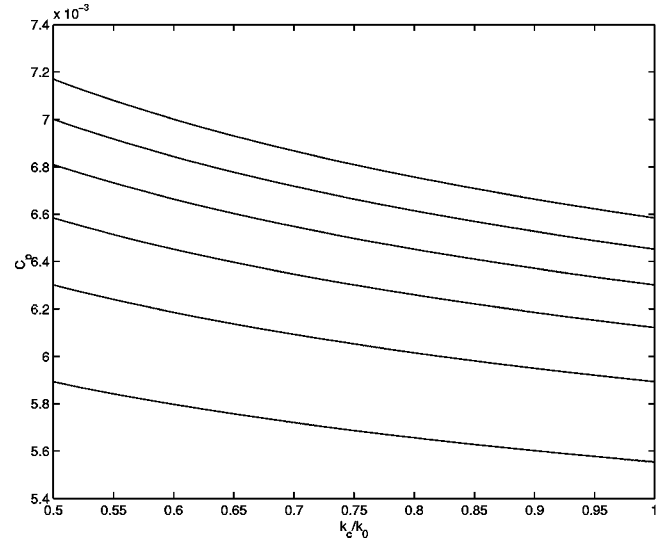

FIG. 9. The behavior of the Smagorinsky constant C_P versus k_c/k_0 for the thermal turbulent transport with $0.001 < k_s/k_0 < 0.006$ and $0.002 < k_p/k_0 < 0.012$; the upper curves correspond the higher ratios.

VII. CONCLUDING REMARKS

In the present study, we have focused on renormalization group analysis of incompressible turbulence, aiming at investigating thermal turbulent transport properties.

First of all, we obtain a recursive relationship for the effective thermal eddy diffusivity $\sigma(k_c)$. The relationship is then applied to determine a formula for the turbulent Prandtl number Pr_t in terms of the turbulent Pecelet number Pe_t. Seeking the connection between Pr_t and Pe_t in a simple formula is of great importance in modeling thermal turbulent flow and has motivated various studies in theoretical analysis and experimental measurement as well as in numerical simulation. The simple algebraic equation (24) with Eq. (21) provides one such connection, whose close resemblance to Eq. (25) obtained by Yakhot *et al.* [12] is remarkable not only because of the similarity in mathematical structure, but also they are derived from entirely different approaches to renormalization group analysis of turbulence. In particular, the present formula has been shown to be in excellent agreement with the results of direct numerical simulations by Bell *et al.* [6] and Kim and Moin [7], and in good agreement with that of Kasagi *et al.* [5]. The formula also shows close agreements with the measured data of Bremhost and Krebs [8] and those adapted from Sheriff and O'Kane [9] and Fuch [10] (but with discrepancy at small Pe_t).

Estimation of the Batchelor constant is then another major effort of study for thermal turbulent transport. Many authors have tried to estimate or measure its value either by theory, experiment, or numerical computation under various thermal flow conditions; the data obtained by different authors scatter in the range from as low as 0.2 of Kraichnan [14] to a value 2.0 of Gurvich and Zubkovsky [22]. What we have done in the present study is to show analytically that the Batchelor constant is not universal and to determine its dependence. For this, we had to pursue the differential version of the recursive relationship for an effective thermal eddy diffusivity. Equation (30) is the required equation which is an inhomogeneous ordinary differential equation and is simple enough to render an exact (invariant) solution which is given by Eq. (33). Also, we have modified Batchelor's scaling laws for the thermal energy spectrum $E_T(k)$ by introducing two connecting functions B_1 and B_2 so that the scaling laws fit better the measured spectra by Boston and Burling [21] for air over sea surface and by Grant *et al.* [20] for water in a tidal channel, but again leaving the proportional constant(s) undetermined. The solution of the invariant effective thermal eddy diffusivity is then employed to determine the Batchelor constant, which was shown to depend on several characteristic wave numbers of the particular flow under consideration.

In spite of the above success, we must recall that the present theory is based on isotropic, stationary, and homogeneous turbulence, and statistical hypotheses (i)–(iii) are not nondebatable. But the close agreement between the results of the present renormalization group analysis and DNS, experiments that typically do not follow the present assumptions about turbulence, should, however, indicate that there must be properties that are universal to all thermal turbulent flows. In particular, the formulas and relationships, as summarized above for various thermal turbulent properties, are simple enough and are amenable to further investigation, verification, and modification under various flow and thermal conditions.

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